Ch6: Lebesgue Measure Theory

from Real Mathematical Analysis

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1 Outer measure

Definition. Lebesgue Outer Measure:
In $\mathbb{R}^1$, define measure of a interval $I := (a, b); |I| = b - a$ and for a set $S \in 2^{\mathbb{R}}$, outer measure

$$m^*(S) = inf\{\sum_{i=1}^{\infty} |I_i| \ s.t. \ \cup_{i=1}^{\infty} I_i \supset S\}$$

Analogously, In $\mathbb{R}^n$, hyper-rectangle $R := (a_1, b_1) \times ...(a_n, b_n)$ and $|R| = (b_1 - a_1) \times ... \times (b_n - a_n)$.and for a set $S \in 2^{\mathbb{R}^n}$, outer measure

$$m^*(S) = inf\{\sum_{i=1}^{\infty} |R_i| \ s.t. \ \cup_{i=1}^{\infty} R_i \supset S\}$$

We have the following properties for outer measure:

Theorem 1.1. Propertied of Lebesgue Outer measure:

- $m^*(\varnothing) = 0$
- $m^*(A) < m^*(B)$ if $A \subset B$
- $A = \cup A_i$, $m^*(A) \leq \sum A_i$

Definition. Zero set: A set $S \in \Omega$, if its outer measure is zero, we call it a zero set.

Proposition 1.1. Countable union of zero set is still measured zero

Proof. Given the permutation of set, and for any number $\epsilon > 0$. Cover $i - th$ set with $\epsilon/2^{i+1}$. Then the outer measur of the set union is smaller than ϵ. □

Theorem 1.2. Bounded Closed box is still the same size as its open counter part in $\mathbb{R}^n$.

Proof. When $n = 1$, and $B = [a, b]$ $m^*(B) \leq b - a$ from $\epsilon-$principle. For the reverse inequality, since B is compact, for all open interval covering, we have a finite subcovering for B. Suppose the subcovering is $\{I_1, ..., I_N\}$. If $N = 1$, $|I_1| > (b - a)$ trivially.

Suppose we know the length sum of a covering by N intervals is bigger than $(b - a)$ for any bounded closed interval, then for a $N + 1$ covering of B, suppose the interval that covers a is $I_1 : (a_1, b_1)$. If $b_1 > b$, $\sum_{i=1}^{N+1} |I_i| \geq (b - a)$ trivially.

Other wise, we have that $B/(a_1, b_1) = [c, b]$ is covered by N interval thus $\sum_{i=1}^{N} |I_{i+1}| \geq (b - c)$and $|I_1| > (a - c)$, so the sum of length is still greater or equal to $b - a$. By induction, the result follows.

When $n > 1$, $m^*(B) \leq \prod_{i=1}^{n}(b_i - a_i) = |B|$ from $\epsilon-$principle. For the reverse inequality, with Lebesuge number lemma, we have a λ s.t. every cube with diameter smaller then λ will fall in a open cube.

Suppose an arbitrary open cubes covering C induce a Lebesgue number λ We may partition on B s.t. every small cube with diameter smaller then λ. Say each small cube is s_i, then we have $\sum |s_i| = |B|$ and $\sum_{s_i \subset C_i} |s_i| \leq |C_i|$. We have that

$$|B| \leq \sum_i \sum_{s_k \in C_i} |s_i| \leq \sum_i |C_i|$$

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2 Measurablility

Definition. Abstract Outer Measure:
Any measure that satisfy Properties of Theorem 1.1 is a Abstract Outer Measure

Definition. Measurable set (Caratheodory's criterion): for any subset A of Ω, $m^*A = m^*A \cap E + m^*A \cap E^c$. Then E is measurable. We call the collection of measurable set (of Ω) $\mathcal{M}$

Example. Non-measurable set:

Definition. $\sigma-$Algebra σ:

1. $\varnothing, \Omega \in \sigma$
2. $A \in \sigma$, $A^c \in \sigma$
3. $E_i \in \sigma, E = \cup E_i, E \in \sigma$

Theorem 2.1. $\sigma(\mathcal{M}) = \mathcal{M}$ with any outer measure. Moreover, the outer measure restricted to this $\sigma-$algebra is countable additivity.

Proof. First of all, proof that $\mathcal{M}$ is a $\sigma-$algebra:

1. $\varnothing, \Omega$ is measurable because $m^*(X) \geq m^*(X \cap \varnothing) + m^*(X \cap \Omega)$ The former one is a zero set, the latter one is a subset of X.

2. If E is measurable, $m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$ for any $X \subset \Omega$. Obviously, E^c is measurable because $(E^c)^c = E$.

3. First, prove that $\mathcal{M}$ is closed under intersection (thus, closed under union, and difference as $\mathcal{M}$ is closed under complement.) Second, prove $\mathcal{M}$ is finite additive if each E_i is disjoint to each other. Third, prove that $\mathcal{M}$ is closed under countable union in disjoint scenario. Finally, prove that $\mathcal{M}$ is countably additive and closed countable under union.

   (a) $A, B \in \mathcal{M}$,
   $m^*X \geq m^*X \cap A + m^*X \cap A^c \geq m^*(X \cap (A \cap B)) + m^*(X \cap (A \cap B^c)) + m^*(X \cap (A^c \cap B)) + m^*(X \cap (A^c \cap B^c)) \geq m^*(X \cap (A \cap B)) + m^*(X \cap (A \cap B)^c)$.
   So it is closed under intersection, and inductively, closed under finite intersection, union, and difference.

   (b) If $\{E_i\}$ are finite and disjoint to each other, we have that $mE = m(E \cap E_1) + m(E \cap E_1^c) = m(E_1) + m(\cup_{i=2}^n E_i)$. Inductively, we have $\{E_i\}$ be additive.

   (c) Suppose $\{E_i\}$ are disjointed to each other, so would $\{E_i \cap X\}$ be. For any n, we have $\cup_{i=1}^n E_i$ measurable and $(\cup_{i=1}^n E_i)^c \supset E^c$ So $m^*X = m^*(X \cap (\cup_{i=1}^n E_i)) + m^*(X \cap (\cup_{i=1}^n E_i)^c) \geq \sum_{i=1}^n m^*(X \cap E_i) + m^*(X \cap E^c)$. Since the $\sum_{i=1}^n m^*(X \cap E_i) + m^*(X \cap E^c)$ increases as $n \to \infty$, we know from monotone convergence theorem, its limit is smaller than m^*X. Then, with the subadditivity property of outer measure, we have the equation:

   $$m^*X = \sum_{i=1}^{\infty} m^*(X \cap E_i) + m^*(X \cap E^c) \tag{1}$$

   (d) Replace X as E in (1), we have the countably additivity. For any countable union of $\{E_i\}$, we may take $\{E_i'\}$ as $E_i' = E_i / \cup_{k=1}^{i-1} E_k$. So we have $E = \cup E_i = \cup E_i'$ with $\{E_i'\}$ disjoints to each other. So we may use (1) to prove that E is measurable.

□

Theorem 2.2 (Measure Continuity Theorem)**.** Suppose $\{E_i\}, \{F_i\}$ are sequence of measurable set.

1. If $E_i \uparrow E$, $m^*(E_i) \to m^*(E)$.

2. If $F_i \downarrow F$, and $m(F_1) < \infty$, $m^*(F_i) \to m^*(F)$

3 Meseomorphism

Definition. Measure Space: a triple $(\Omega, \mathcal{F}, \mu)$ is a measure space if Ω is a set, $\mathcal{F}$ is the $\sigma-$algebra of some subsets of Ω, and μ is a measure.

Note that don't confuse with measurable space $(\Omega, \mathcal{F})$ which does not require a measure.

Now suppose we have two measure space $(\Omega, \mathcal{F}, \mu), (\Omega', \mathcal{F}', \mu')$.

Definition. For a $T : \Omega \to \Omega'$, it is:

1. Mesemorphism, if $X \in \mathcal{F}$, $TX \in \mathcal{F}'$.
2. Meseomorphism, if T is a bijection of mesemorphism.
3. Mesisometry: if $\mu'(TX) = \mu(X)$.

Theorem 3.1. Suppose T is a bijection with $\mu'^*(TX) \leq t\mu^*(X)$ and $\mu^*(T^{-1}X') \leq t^{-1}\mu'^*(X')$. It is a mesomorphism.

Proof. First, prove the inequality is actually equation. Second, use an arbitrary test set to prove the mesomorphism property.

1. $\mu^*(X) = \mu^*(T^{-1}(T(X))) \leq t^{-1}\mu'^*(T(X)) \leq t^{-1}t\mu^*(x) = \mu(X)$ So the equation holds.
2. For an arbitrary test set $X \subset \Omega$, $TX = X' \subset \Omega'$. Also, from the fact that T is bijection, $T(A \cap B) = TA \cap TB$.
   $\mu'^*(X') = \mu'^*(TX) = t(\mu^*(X)) = t(\mu^*(X \cap E) + \mu(X \cap E^c)) = t(t^{-1}(\mu'^*(T(X \cap E)) + \mu'^*(T(X \cap E^c))) = \mu'^*(X' \cap TE) + \mu'^*(X' \cap TE^c)$. So TE is also measurable.

□

4 Regularity

Here the mesaure theory focus on $\mathbb{R}^n$. Now we let the Lebesgue measure as m.

Theorem 4.1. All open sets are Lebesgue measurable.

Proof. Form 4.1.1,we have that all open sets is in $\sigma(\{\text{half space}\})$. The result follows. □

Lemma 4.1.1. All half spaces $H = (a, \infty) \times \mathbb{R}^{n-1}$ in $\mathbb{R}^n$ are measurable.

Proof. Set the test set $X \in \mathbb{R}^n$ and a open half space $(a, \infty) \times \mathbb{R}^{n-1}$. We can find a countable cube covering $\{R_i\}$ that covers X with $\sum |R_i| < m^*(X) + \epsilon$.

For each R_i, cut it into $R_i^+ := R_i \cap H$ and $R_i^- := R_i \cap H^c$. We have $\cup|R_i^+| \supset X \cap H$ and $\cup|R_i^+| \supset X \cap H$. Consequently, $m^*(X \cap H) \in \cup R_i^+$ and $m^*(X \cap H^c) \in \cup R_i^-$ Moreover, $\sum |R_i^+| + |R_i^-| = \sum |R_i|$.

Consequently, $m^*X \leq m^*(X \cap H) + m^*(X \cap H^c) \leq \sum |R_i^+| + |R_i^-| = \sum |R_i| \leq m^*X + \epsilon$.

Since the ϵ is arbitrary, we have the measuribility of H. □

Definition. F_σ and G_δ set:

1. F_σ is the collection of countable union closed set.
2. G_δ is the collection of countable intersection of open set.

Theorem 4.2. The regular property of Lebesgue measure:

A set E is measurable if and only exist $F \in F_\sigma, G \in G_\delta$ such that $F \subset S \subset G$ and $m(G/F) = 0$.

Proof. For the necessary direction, we have $E = G \cup E \setminus G$. $m^*(E \setminus G) = 0$ implies that $E \setminus G$ is a measurable set. So E is measurable.

For the sufficient direction, with the 4.2.1, we know the result hold if E is bounded.

If E is unbounded, let $E_i = (R_i \setminus R_{i-1}) \cap E$ with R_i the cube of side length $2i$, centred at 0.

Pick the open covering U_n^i that covers the E_i and $m(U_n^i) < m(E_i) + \frac{1}{n2^i}$. From the fact that $\cup_{i=1}^{\infty} U_n^i \setminus E \subset \cup_{i=1}^{\infty}(U_n^i \setminus E_i)$ We know $m(\cup_{i=1}^{\infty} U_n^i \setminus E) \leq m(\cup_{i=1}^{\infty}(U_n^i \setminus E_i)) \leq \sum_{i=1}^{n} m(U_n^i) - m(E_i) \leq \frac{1}{n}$

For each E_i, take out the G_δ set V_i, because $m(E_i) = m(V_i)$ we have $m(\cup_{i=1}^{\infty} U_n^i \setminus \cup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{n} m(U_n^i) - m(V_i) \leq \frac{1}{n}$

So let $U_n = \cup_{i=1}^{\infty} U_n^i, \cap U_n = U$, we have $m(U \setminus V) = 0$. Obvious, U is F_σ set and V is G_δ set and $U \supset E \supset V$. □

Lemma 4.2.1. Regularity sandwich:

A bounded set E is measurable if and only if it has a regular sandwich $F \in F_\sigma, G \in G_\delta$, such that $F \subset E \subset G$ and $m(G) = m(F)$.

Proof. For the sufficient direction, take a rectangle R contains E and let $E^c = R \setminus E$. We have $mR = mE + mE^c$. So we have some open set U_n, V_n s.t. $U_n \downarrow U, V_n \downarrow V$ with $\forall U_n \supset E, V_n \supset E^c$, and $mU_n \to mE$, $mV_n \to mE^c$.

With this we already have $\cap U_n = U, \cap V_n = V$ and they are F_σ sets and $mU = mE$. Then take $V_i' = V_n^c \cap R$, $\cup V_i' = R \cap (\cup V_n)^c = m(R \cup V_n) = m(R) - m(E^c) = m(E)$, we have $V_i' \in E$, closed, and $\cup V_i'$ is a G_δ set that $m(\cup V_i') = m(E)$.

Consequently, the result follows.

For the necessary direction, Because $F \supset E \supset G$, we have $m^*(E \setminus G) \leq m^*(F \setminus G) = 0$, so $E \setminus G$ is a measurable set. From the fact that G is measurable set, the result follows.

□

Corollary 4.2.1. Lipeomorphism (Lipschitz continuous, and bijection) is a mesomorphism

Proof. By definiton, Lipschitz continuous function map each set E to $f(E)$ with $m(f(E)) \leq t(m(E))$. So it maps zero set to zero set. Consequently, the regular sandwich relation for any $G \subset E \subset F$ still holds. □

4.1 Affine motion

Theorem 4.3. An affine motion $T : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism and mutiplies the measure by $|\det T|$

Since every linear transformation can be decomposed as $O_1 D O_2$ with O_1, O_2 orthonormal (Polar Decomposition), From the lemma 4.3.1, 4.3.2, For any measurable set, we can write it as $\cup B_i \cup Z_1$ and $\cup C_i \cup Z_2$ with Z_1, Z_2 zero set ,and B_i and C are open disjointed balls and cubs.

O_1, O_2 maps each B_i to another B_i' with the radius the same and still disjointed to each other.

D maps each C_i to size of $|\det T| C_i$, and still disjointed to each other.

Moreover, D,O_i maps zero set to zero set as they are Lipschitz. Consequently, T is a meseomorphism and maps $m(T(E)) = |\det T| m(E)$.

Lemma 4.3.1. Every open set in R^n is a countable union of disjoint open cubes plus a zero set.

Lemma 4.3.2. Every open set in R^n is a countable union of disjoint open balls plus a zero set.

Proof. This is not the point of the chapter, so neglect it now. □

4.2 Hull, Kernel, Inner Measure

Definition. Hull and Kernel, (Measure theoretic) Boundary of a set E:

1. Hull: The smallest G_δ set that contains E
2. Kernel: The biggest F_σ set contained in E
3. (Measure theoretic) Boundary: $H_E \setminus K_E$

Definition. Inner measure m_*: which is measure of the kernel of a given set.

Theorem 4.4. Measurability of a set in a box: $A \subset B \subset \mathbb{R}^n$ with B a box, we have $m^*B = m^*A + m^*(B \setminus A)$ if and only if A is measurable.

Proof. The necessary direction simply follows from the Caratheodory definition.

For any $K \subset A$ is closed, We have $B \setminus K$ is open and contains $B \setminus A$. Also, $mB = mK + m(B \setminus K)$. Then, take $K \to K_A$, we have $mB = m_*A + m^*(B \setminus A)$.

From the conditions, we have $m_*A = m^*A$, so A is measurable. □

5 Products and Slices

Here, we merely consider in $\mathbb{R}^n$ space.

Theorem 5.1. Measurable Product Theorem:
If $A \in \mathbb{R}^n, B \in \mathbb{R}^m$ are Lebesgue measurable, Then, $m(A \times B) = m(A)m(B)$. Let $0 \cdot \infty = 0$ for convenience.

Proof. From 5.1.3, and the $\sigma-$property of measurablility, we have every F_σ, G_δ set, $m(A \times B) = m(A) \times m(B)$.

Then, if A, B measurable, take F_A, G_A, F_B, G_B the F_σ, G_δ set with $F_A \supset A \supset G_A, m(F_A) = m(G_A)$ and $F_B \supset B \supset G_B, m(F_B) = m(G_B)$. Obviously, $F_A \times F_B$ is still a F_σ and $G_A \times G_B$ is still a G_δ.

Consequently, $F_A \times F_B \supset A \times B \supset G_A \times G_B$ and $m(F_A \times F_B) = m(A \times B) = m(G_A \times G_B) = m(A)m(B)$ □

Lemma 5.1.1. For product of cubes: $m(A \times B) = m(A)m(B)$

Proof. This has been deduced previously. □

Lemma 5.1.2. For product with a zero set: $m(A \times Z) = 0$

Proof. We may use ϵ method to covers Z with countable union of cubes $\cup C$ with total measure smaller than ϵ. Use a big cube R to cover A if A is bounded. So we have $A \times Z \leq R \times \cup C = 0$.

For A unbounded case, we may use R_i to approach A. The result follows. □

Lemma 5.1.3. For product of open sets: $m(A \times B) = m(A)m(B)$

Proof. Because Each open set can be written as countable disjointed union of cubes plus a zero set, and multiply of zero set with any set is still measure zero.

Pick $A = \cup_{i=1}^{\infty} C_i^a \cup Z_a$, $B = \cup_{i=1}^{\infty} C_i^b \cup Z_b$.

So $m(A \times B) = m(\cup_{i,j \in \mathbb{N}} C_i^a \times C_j^b + Z) = m(A) \times m(B)$ □

Definition. Slice: for a set $E \subset \mathbb{R}^n \times R^m$, the slice of E on $x \in \mathbb{R}^n$ is

$$E_x = \{y \in \mathbb{R}^m | (x, y) \in E\}$$

Theorem 5.2. Quasi-Chebyshev theorem:

Suppose $W \in I^{n+m}$ is open, $\alpha > 0$. Take $X_\alpha := \{x \in \mathbb{R}^n | m(W_x) > \alpha\}$. Then,

$$m(W) \geq \alpha\ m(X_\alpha)$$

Proof. The openess of W gave us that every slice of W is open. Pick $x \in X_\alpha$, we have a compact set K_x with $m(K_x) > \alpha$. We may find a open set around $x = U(x)$ with $x' \in U(x), W_{x'} \supset K_x$ from the fact that W is open. This gave us that X_α is a open set in $\mathbb{R}^n$, thus it can be written as $\cup_{i=1}^{\infty} I_i$ with each I_i an open cube and in each I_i, contain an K_i such that $\forall x \in I_i, W_x \supset K_i$ with $m(K_i) > \alpha$.

For each compact set K in X_α, we may reduce the X_α (which covers K) to $\cup_{i=1}^{n} I_{j_i}$, which can be cut into finite many disjoint open cubes with a zero set.

Thus, we have

$$m(W) > m(\cup_{i=1}^{n} I_{j_i} \times K_{j_i}) = \sum_{i=1}^{n} m(I_{j_i})\ m(K_{j_i}) > \alpha \sum_{i=1}^{n} m(I_{j_i}) > \alpha m(K)$$

So take $K \to X_\alpha$, we has the equality still holds and by $m_*(X_\alpha) = m(X_\alpha)$, the result follows. □

Theorem 5.3. Zero Slice Theorem:

E is a zero set if and only if almost every slice of E is a zero set.

Proof. The sufficient direction can be approached by 5.2 simply. As non measure zero slice is $\cup_{n=1}^{\infty} X_{\frac{1}{n}}$, since each $X_{\frac{1}{n}}$ is measure zero, so is its union.

For the necessary direction, we may proof the $E_x = 0\ \forall x \in \mathbb{R}^n$, and $E \subset I^n$ case first.

Take a compact set $K \in E$, and the slice of x on K_x can be cover by a open set V with $m(V) \leq \epsilon$. By the compactness of K, we know there is a open ball U around x such that $\forall x' \in U$, $K_{x'} \in V$.

Consequently, we can covers K by $\cup_{x \in R^n} U(x) \times V(x)$, and we can pick those finite $U(x) \times V(x)$ that covers K.

Then, we can construct $U'(x)$ the disjoint set from $U(x)$, so $m(\cup U'(x) \times V(x)) \leq 1 \cdot \epsilon$.

With ϵ method, generalise this to unbounded set. Also, it is trivial to prove zero set with slice on them is non-zero. □

6 Lebesgue Integral

We first take $f : \mathbb{R}^n \to \mathbb{R}^+$. For simplicity, we may use $n = 1$ in most of the time.

Definition. Undergraph and Completed graph of f:

1. Undergraph of f:
$$\mathcal{U}(f) = \{(x, y) \in \mathbb{R}^{n+1} | 0 \le y < f(x)\}$$

2. Completed of f:
$$\hat{\mathcal{U}}(f) = \{(x, y) \in \mathbb{R}^{n+1} | 0 \le y \le f(x)\}$$

Definition. (Lebesgue) Measurable Function f: f is a measurable function if and only if $\mathcal{U}f$ is measurable in $\mathbb{R}^{n+1}$.

Also, we say f is Lebesgue integrable if $\mathcal{U}f < \infty$ and write $\mathcal{U}f = \int f$.

Here we do not the dx in Riemann sense because we want to emphasis that it is the Lebesgue measure of undergraph.

When we say $f_n \to_{a.e.} f$, it means that almost every point in Domain of f_n converge to f.

Theorem 6.1. Monotone Convergence Theorem: If $f_n \uparrow_{a.e.} f$ and every f_n is measurable, We have f measurable, and $\int f_n \to \int f$.

Proof. This is simply an application of measure continuity theorem. □

Definition. Lower and Upper envelope sequence: For f_n be a sequence of functions.

1. Lower envelop $\underline{f}_n := \inf_{k \ge n} f_k$

2. Upper envelop $\bar{f}_n := \sup_{k \ge n} f_k$

Obviously, $\bar{f}_n$ is decreasing and $\underline{f}_n$ is increasing as $n \to \infty$.

With this we have

$$\int \bar{f}_n = m(\cup_{k=n}^{\infty} \mathcal{U}(f_k)), \text{and } \int \lim_{n\to\infty} \bar{f}_n = m(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \mathcal{U}(f_k))$$

And

$$\int \underline{f}_n = m(\cap_{k=n}^{\infty} \mathcal{U}(f_k)), \text{and } \int \lim_{n\to\infty} \underline{f}_n = m(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \mathcal{U}(f_k))$$

Theorem 6.2. Fatou's Lemma:

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

Proof. Because $\underline{f}_n$ is increasing, with MCT, we know $\int \lim_{n\to\infty} \underline{f}_n = \lim_{n\to\infty} \int \underline{f}_n$ and $\underline{f}_n \uparrow f$ to some measurable f.

For each f_n, we have $f_k \ge \underline{f}_n\ \forall k \ge n$, so $\inf_{n\to\infty} \int f_n \ge \int \underline{f}_n$, the inequality still holds when we take limit. □

Corollary 6.2.1 (Reverse Fatou)**.** Suppose $f_n < g$ and g is integrable. We have $\limsup \int f_n \leq \int \limsup f_n$.

This can be simply deduced as Fatou.

Theorem 6.3. (Lebesgue) Dominant Convergence Theorem:

If $f_n \to f$ pointwisely, and each f_n is bounded by an integrable function g, $\int f_n \to \int f$

Proof. The convergence of f_n gave us that $\liminf f_n = \limsup f_n$, so $\int \liminf f_n = \int \limsup f_n$ From Fatou's lemma and reverse Fatou, we know

$$\int \liminf f_n \leq \liminf \int f_n \leq \limsup \int f_n \leq \int \limsup f_n$$

The boundeness of f_n guarantees the last inequality holds, and the $\int \liminf f_n = \int \limsup f_n$ let us know equation holds everywhere. □

Definition. f-translation T_f: $T_f(x, y) = (x, y + f(x))$.

Theorem 6.4. If f is integrable T_f is a mesiometry.

Proof. When f is an step function, it would be trivial. Also, we have $\mathcal{U}f \cup T_f(\mathcal{U}g) = \mathcal{U}(f+g) = T_g(\mathcal{U}f) + \mathcal{U}g$.

Then for each cube K^{n+1}, we can construct an $g = \chi_{K^n}$, than, $m(\mathcal{U}f) + m(T_f(K^{n+1})) = m(T_g(\mathcal{U}f)) + m(K^{n+1})$.

So $m(K^{n+1}) = m(T_f(K^{n+1}))$. Since every measurable set can be sandwich by G_δ and F_σ, the result follows. □

7 Italian Measure Theory

Although in Lebesgue Integral, we do not write dx as the differential term, we may still write $\int f dx$ to indicate the integration variable.

Proposition 7.1. Cavalieri's Principle: Suppose the $E \in \mathbb{R}^{n+m}$ is measurable, $x \in \mathbb{R}^n, E_x$ is measurable a.e., and the function $x \to m(E_x)$ is also measurable. Moreover,

$$m(E) = \int m(E_x) dx$$

Proof. This holds true when E is zero set or cube, so it holds true of any open set.

Consequently, it still hold true for every bounded set, with ϵ-method, this can be generalised to every measurable set. □

Theorem 7.1. Preimage definition of measurable function is equivalent to Undergraph definition of measurable function.

Proof. From Preimage definition to Undergraph definition can be deduced by characteristic function and monotone.

From Undergraph to Preimage: From Cavalieri's Principle, $(\mathcal{U}f)_y$ is measurable almost everywhere, and it is obvious that $(\mathcal{U}f)_y = \{x | f(x) \geq y\}$. Obviously, $(\mathcal{U}f)_y \supset (\mathcal{U}f)_{y'}$ if $y \leq y'$ So we may choose an $y_i \downarrow y$ with $(\mathcal{U}f)_{y'}$ measurable. By measure continuity theorem, the result follows. □

Theorem 7.2. Fubini's theorem: $f(x,y) \to R$ is measurable,then

$$\int\int f_x(y)dydx = \int\int f_y(x)dxdy = \int f$$

Proof. This is trivial from Cavalieri's Principle. □

8 Vitali Coverings and Density Points

Definition. Vitali Covering:

For a set S and $\mathcal{V}$ a covering of S, if $\forall \epsilon > 0, p \in S$, $\exists V \in \mathcal{V}$ s.t. $p \in V, diam(V) \leq \epsilon$. $diam(.)$ is the diameter of a given set.

Theorem 8.1. Vitali's Covering Theorem:

If $\mathcal{V}$ is a closed ball Vitali Covering of set S, exist a countable subcollection of $\mathcal{V}$, says $\cup_{i=1}^{\infty} V_i = U$ s.t.

1. V_i, V_j disjoint to each other.
2. $\sum_{i=1}^{\infty} m(V_i) < m^*(S) + \epsilon$
3. $m^*(S \setminus U) = 0$

We call this U covers S efficiently (almost every S).

Proof. Firstly assume S is bounded. Covers S with a open set W with $m(S) < m^*(S) + \epsilon$, retake $\mathcal{V} = W \cap \mathcal{V}_0$ with $\mathcal{V}_0$ the original Vitali covering. We have $\sup_{V \in \mathcal{V}} diam(V) \leq \infty$ now. Then, construct a sequence of U similar to 8.1.1, it is the desired efficient covering.

It must satisfy (1),(2) for sure. Take $U_n = \cup_{i=1}^{n} B_i$, it is closed , obviously. So we have $\{B \in \mathcal{V}, B \cap U_n = \varnothing\}$ still a Vitali covering that covers $S \setminus U_n$. Moreover, We have $\cup_n^{\infty} B_i$ a collection of $\cup_{i=n}^{\infty} 5B_i \supset \{B \in \mathcal{V}, B \cap U_n = \varnothing\} \supset S$. So $\cup_{i=n}^{\infty} 5m(B_i) > m^*(A \setminus U_n)$.

With $\cup_{i=n}^{\infty} 5m(B_i) \to 0$, (3) holds spontaneously.

For the unbounded case, approaching it from bounded subspace, the results follows. □

Lemma 8.1.1. Vitali's Covering Lemma:

In a separable metric space, for any collection of closed balls $F = \{B_i | i \in J\}$ with $\sup\{diam(B_i) | i \in J\} \leq \infty$, we can find a countable disjointed collection of balls $\cup_{i=1}^{\infty} B_{j_i}$ with

$$\cup_{i=1}^{\infty} 5B_{j_i} \supset \cup_{i \in J} B_i.$$

$5B$ means B still centred in the same place, only the diameter expends five times.

Proof. Firstly, suppose F is bounded.

Set $R_0 = \sup_{B \in H} diam(B)$, pick B_{j_1} with $diam(B_{j_1}) > \frac{1}{2} R_0$

Iteratively, let $H_i = \{B | B \in H_{i-1},\ B \cap \cup_{k=1}^{i-1} B_{j_k} = \varnothing\}$, B_{j_i} with $diam(B_{j_i}) > \frac{1}{2} \sup_{B \in H_{i-1}} diam(B)$.

Collect each B_{j_i} and construct $U = \cup_{i=1}^{\infty} B_{j_i}$, if $B \cap U \neq \varnothing$ we have $B \subset 5U$ simply from triangular inequality.

Because F is bounded, we have $diam(B_{j_i}) \to 0$. Thus, for each $B \in F diam(B) > 0$, it must fall out from some H_n.

Bounded cased is proved.

Fro the unbounded case, We can approach the unbounded F by countable increasing closed set. The result follows. □

8.1 Density Point

Definition. Density:

We say the concentration of measurable set E in Q is

$$\frac{m(E \cap Q)}{m(Q)}$$

Or $[E : Q]$ for simplicity

So we say the density of E in p, a point in E is

$$\lim_{Q \to p} [E : Q]$$

Or $\delta(E, p)$ for simplicity, and $\bar{\delta}, \underline{\delta}$ as limit inf and limit sup.

If $\delta = 1$ we say the point is a density point.

Theorem 8.2. Lebesgue Density Theorem:

If E is measurable, $\delta(p, E) = 1$ for almost every $p \in E$.

Proof. Fix an $1 > a > 0$. Define $X_a = \{x | x \in E, \underline{\delta}(x, E) < a\}$. It means for every $p \in X_a$, we can find a closed cube Q such that $x \in Q$, $|Q| < \epsilon$, and $[E : Q] < a$. Collect all these Q. Obviously, it is a Vitali Covering.

By VCT, we may have an $\cup_{\mathbb{N}} Q_i$ that covers X_a efficiently, then we have

$$\begin{aligned} m^*(X_a) &< \sum m(Q_i) + \epsilon \\ &= \sum m(Q_i \cap X_a) + \epsilon \\ &\le a \sum m(Q_i) \le a(m^*(X_a) + \epsilon) \end{aligned}$$

Since ϵ and a are arbitrary, the result follows. □

9 Lebesgue Calculus

Definition. Average and density of a function:

Average of f on a measurable set A is

$$⨍_A f = \frac{1}{m(A)} \int_A f = [f : A]$$

.

Density of f on a point p is

$$\delta(p, f) = \lim_{Q \downarrow p} [f : Q]$$

Theorem 9.1. Average Value Theorem:

Take a locally integrable function f,

$$f(p) = \delta(p, f)$$

almost every p in domain.

Proof. WOLG, We may refrain f on an interval X and assume $f > 0$. Given $\alpha > 0$, $I_k = [k\alpha, (k+1)\alpha)$, $I_k^{-1} = f_k^{-1}$. Suppose $f(p) \in I_k$, Write

$$\fint_Q f = \frac{1}{m(Q)}(\int_{A\cap Q} f + \int_{B\cup Q} f + \int_{C\cup Q} f)$$

with $A = \cup_{i=0}^{k-1} I_i^{-1}$, $B = I_k^{-1}$, $C = \cup_{i>k} I_i^{-1}$.

With $[A : Q] \to 0$ and $[B : Q] \to 1$, and f is bounded in A, B, we have $[A : Q](\fint_{A\cap Q} f) + [B : Q](\fint_{B\cup Q} f)$ bounded in $[k\alpha, (k+1)\alpha]$ when $m(Q) \to 0$.

For the third term, truncate f with $f_n = \min(f, n)$, and $g_n = f - f_n$. Because f is integrable, we know $\int g_n \to 0$. Pick

$$X(\alpha, g_n) = \{x|\bar{\delta}(x, g_n) > \alpha\}$$

It is easily to see that $m(X(\alpha, g_n)) \to 0$ from 9.1.1. So $X(\alpha, g_n)^c = X$ a.e.

That is, for almost every p, we may find an n s.t. $\bar{\delta}(p, g_n) \leq \alpha$.

So the third term is

$$\frac{1}{mQ}\int_{C\cap Q} f_n + \frac{1}{mQ}\int_{C\cap Q} g_n$$

The first one tends to 0 as $[C : Q] \to 0$, the second one is smaller than α.

Combine all of three terms, we have $[f : Q] \in [k\alpha, (k+2)\alpha]$ for almost every p. With $\alpha \to 0$, the result follows. □

Lemma 9.1.1. Chevyshev's Density inequality:

Define $X_{a,f} = \{x : \bar{\delta}(x, f) > a\}$,

$$a \cdot m(X_{a,f}) \leq \int f.$$

Proof. For each $x \in X_{a,f}$, we have a small Q covers x and $[f : Q] > a$. Collect these Q, its a Vitali covering. We may find a efficient covering $V = \cup_{\mathbb{N}} Q_i$ covers $X_{a,f}$ with $a \leq [f : Q_i] \implies a \cdot m(Q_i) \leq \int_{Q_i} f$.

So

$$a \cdot m^*(X_{a,f}) \leq \sum a \cdot m(Q_i) \leq \sum_{\mathbb{N}} \int_{Q_i} f \leq \int f$$

The result follows. □

Corollary 9.1.1. *Assume a $f : [a, b] \to \mathbb{R}$ is integrable. Take $F(x) = \int_a^x f(x)$, we have $F'(x) = f(x)$ a.e.*

Proof. From the Average Value Theorem, we have Q is actually an interval, so

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \fint_h f(t) = f(x)$$

Same for $[x - h, h]$. □

Definition. Absolutely Continuous: For every $\epsilon > 0$, exist an $\delta > 0$ s.t. If $\sum_{i=1}^{n} |[a_i, b_i]| < \delta$, $\sum_{i=1}^{n} |f(a_i - b_i)| < \epsilon$, and $[a_i, b_i]$ disjoints to each other.

Measure Continuous: If Z is a zero set, $m(f(Z)) = 0$.

Theorem 9.2. Lebesgue's Fundamental Theorem:

Take $f : [a, b] \to \mathbb{R}$ integrable, and $F(x) = \int_a^x f(t)dt$, then:

1. F is absolutely continuous.
2. $F' = f$ a.e.
3. If G absolutely continuous, and $G' = f$, $G - F = c$.

Proof. Assume $f > 0$ WLOG.

1. If f is bounded in M, it is obvious that $\sum m(F(I_k)) \leq \sum Mm(I_k)$. For any *epsilon* > 0, we may pick $\delta = \frac{\epsilon}{M}$, so F is absolutely continuous.

   If not, we can chop f into $f_n = f\chi_{f(x)<n}$ and $g_n = f - f_n$. It is obvious that $f_n \to f$ and $\int g_n \to 0$.

   For each ϵ, pick n such that $\int g_n < \epsilon/2$.

   We may find a δ for the f_n function that satisfies the absolutely continuous condition on $\epsilon/2$. Then, for the disjointed intervals I_k with sum less than ϵ,

$$\sum m(F(I_i)) = \sum \int_{I_i} f_n + g_n \leq \sum \int_{I_i} f_n + \int g_n \leq \epsilon$$

2. This is Corollary 9.1.1
3. Take $H = G - F$, so H is still absolutely continuous and $H' =_{a.e.} 0$.

   Pick a fixed $x' \in [a, b]$, and define $X = \{x \in [a, x'] | H'(x) = 0\}$. Fix a $\epsilon > 0$. For every $x \in X$, we have

$$\frac{H(X + h) - H(x)}{h} \leq \frac{\epsilon}{2(b - a)}$$

   with small h (obviously $x \in [x, x + h]$). This forms a Vitali covering. We may find an $\cup I_i$ that covers X efficiently. and some N that $\sum_{i=1}^{N} |I_i| > (x' - a - r)$ with $r > 0$

   With the same ϵ, we may find a δ satisfy the absolutely continuous condition of H in $\epsilon/2$. Pick $r = \delta$, and J is the collection of $[a, x'] - I$, obvious, is still a finite collection of interval. We have

$$H(x') - H(x) = \sum_{i=1}^{N} H(I_i) + \sum_{i=1}^{|J|} H(J_i) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$$

   The former comes from the $\frac{H(X+h)-H(x)}{h} \leq \frac{\epsilon}{2(b-a)}$, and the latter comes from absolutely continuous property.

□

10 Lebesgue's Last Theorem

Theorem 10.1. A monotone function f in $[a, b]$ is differentiable almost everywhere.

Proof. Take $D^+_M f(x)$ as the $\limsup_{h\to 0} \frac{f(x+h)-f(x)}{h}$, and similar for $+, -$, and M, m.

Define $E_{sS} = \{x | D^-_m f(x) < s < S < D^+_M f(x)\}$ with some $s < S$.

Since for every $x \in E_{sS}$ it is contained in some small $[x-h, h]$ such that $\frac{f(x)-f(x-h)}{h} < s$. We may has an efficient covering $L = \cup_{\mathbb{N}}[a_i, b_i]$that covers E_{sS} with $\frac{f(a)-f(b)}{a-b} < s$ for each interval. Similar for the R which is another efficient covering that $\frac{f(a)-f(b)}{a-b} > S$, and moreover, $R_i \in L_j$ for some j. Thus, with Lemma 10.1.1,

$$m^*(E_{sS}) \le m(R) = \sum_i \sum_{R_j \in L_i} |R_j| \le \sum_j \frac{s}{S}|L_j| \le (\frac{s}{S} m^*(E_{sS})) + \epsilon$$

. So E_{sS} is zero set for any s, S. Change $+, -$ and M, m in the same way. We proved that derivative existed a.e.

In addition, we may prove the $f'(x)$ is finite a.e. Define $g_n(x) = n(f(x+\frac{1}{n}) - f(x))$. g_n is measurable and $g_n \to f'$ a.e., so f' is measurable, too.

Then, $\int_a^b f' = \int_a^b \liminf_{n\to\infty} g_n \le \liminf \int_a^b g_n$,

We have $\int_a^b g_n = n\int_b^{b+1/n} f - n\int_a^{a+1/n} f$(take $f = f(b)$ if $x > b$.) The first one is $f(b)$, and the second one is bigger than $f(a)$. Combine we have $\int_a^b g_n \le f(b) - f(a)$. The result follows. ☐

Lemma 10.1.1. Chebyshev's Lemma:

If f is monotonely increasing on $[a, b]$, $\frac{f(b)-f(a)}{b-a} = s$ take $I = \{[a', b'] \subset [a, b] | \frac{f(b')-f(a')}{b'-a'} > S\}$, $S > s$ and each interval in I is disjointed. Then we have

$$|I| \le \frac{s}{S}(b-a)$$

Proof. Because f is nondecreasing, $s(b-a) = (f(b)-f(a)) \ge \sum_{[a'_k, b'_k]\in I} f(b'_k) - f(a'_k) \ge \sum_{[a'_k, b'_k]\in I} S(b'_k - a'_k)$. ☐

Corollary 10.1.1. Lipschitz function is differentiable a.e.

Definition. Bounded Variation of a function f on S:

For every partition P of S: $\sum_P \Delta f < C$ with C is a constant. We say P is of B.V.

Theorem 10.2. An absolutely continuous function on $[a, b]$ is of B.V.

Proof. We may find an $\delta > 0$ such that $\sum_{i=1}^\infty |b_i - a_i| \le \delta \implies \sum_{i=1}^\infty |f(b_i) - f(a_i)| < 1$.

We may dissect $[a, b] = \cup_{k=0}^M [a + k\delta, a + (k+1)\delta]$. So in each interval, the B.V. is smaller than 1, the total B.V, as a result, smaller than M. ☐

Corollary 10.2.1. An function f of B.V is differentiable a.e.

Proof. We may write f as subtraction of two increasing function. (How?) Because increasing function is differentiable a.e., so is their subtraction. ☐

Theorem 10.3. Lebesgue's main theorem:

Lebesgue's fundamental theorem is the if and only if relation.

11 Additional Topics

Here we talk about some other things I am lazy to categorise.

11.1 Littlewood's Three Principles

Littlewood introduced the concept of "nearly". which meas except an ϵ set with $\epsilon > 0$.

Theorem 11.1. Littlewood's First Theorem:
For every measurable set, it contains an compact set that is nearly the set.

Which is the regularity of measurable set.

Theorem 11.2. Littlewood's Second Theorem:
Every measurable function is nearly continuous.

Proof. The codomain of the function f can be covered by rational endpoints $I_i = (q_j, q_l)$, which is countable and take $E_i = \{f \in I_i\}$, which is sandwiched by some closed and open sets $K_i \subset E_i \subset U_i$ with $m(U_i \setminus K_i) \leq \epsilon/2^i$. Take $S_i = U_i \setminus K_i$, so $m(\cup S_i) \leq \epsilon$ and define $T = (\cup S_i)^c$.

Suppose $\forall x_k \in T$, $x_k \to x \in T$, fixed an $\sigma > 0$. We must have some $|I_j| \leq \sigma$ and $f(x) \in I_j$, and obviously, $x \in E_j \subset U_j$. From the openess of U_j we know for some K, $x_k \in U_j$ if $k > K$. Moreover, because x_k not in S_i, it must be in K_i, too. $x_k \in E_j$ as well. Consequently, $f(x_k) \in E_j$, and $|f(x_k) - f(x)| \leq \sigma$. □

Theorem 11.3. Littlewood's Third Theorem:
Almost everywhere convergence (of measurable function on a compact interval $[a, b]$) implies nearly convergence.

Proof. $f_n \to f$ a.e. imply for every l, define $X_{k,l} = \{x||f_k > f| > \frac{1}{l}\}$, $X_{k,l} \to_{a.e.} [a, b]$. as $k \to \infty$

Fix an $\epsilon > 0$, we can construct a sequence $X_{k(l),l}$ with $m(X^c_{k(l),l}) \leq \frac{\epsilon}{2^l}$. Take $l \in \mathbb{N}$, we have $m(\cup X^c_{k(l),l}) < \epsilon$ so $(\cup X^c_{k(l),l})^c = \cap X_{k(l),l}$ differ from $[a, b]$ with only an ϵ set.

In the $\cap X_{k(l),l}$, for any given $\sigma > 0$, we may find an $1/l < \sigma$, so for all $x \in X_{k(l),l}$, $|f_n(x) - f(x)| \leq 1/l < \sigma$.

Because $X_{k(l),l} \supset X_{k(l),l}$, the result follows. □

11.2 L^p spaces

Definition. L^p norm of a function f is $(\int |f|^p)^{-p} = ||f||_p$.
If $p = \infty$, we have $||f||_\infty = \inf c||f| < c\ a.e.$

Theorem 11.4. Holder's inequality:
$||fg||_1 < ||f||_p||g||_q$ with $\frac{1}{p} + \frac{1}{q} = 1$

Proof. Young's Inequality: $ab \leq \frac{a^p}{p}\frac{b^q}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$

Take $a = |f|/(||f||_p)^{1/p}$, $b = |g|/(||g||_q)^{1/q}$, so

$$\int ab \leq \frac{1}{p}\int a^p + \frac{1}{q}\int b^q = 1$$

.

Replace a, b with original form, the result follows. □

Because I use LaTeX, I am too lazy to use the o with two dots.

Theorem 11.5. Minkowski's Inequality:

$$||f + g||_p \leq ||f||_p + ||g||_p$$

with $p > 1$.

Proof.

$$\begin{aligned} ||f+g||_p^p = \int |f+g|^p \leq \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\ \leq (||f||_p + ||g||_p)(\int |f+g|^{(p-1)\frac{p}{p-1}})^{\frac{p-1}{p}} \text{(with Holder)} \\ \leq (||f||_p + ||g||_p)(||f+q||_p^{p-1}) \end{aligned}$$

Cancel two side and get the answer. □

Theorem 11.6. For $f_n \to f$ in L^p, there's a subsequence such that $f_{n_k} \to_{a.e.} f$

Proof. Converge in L^p implies converge in measure by Chebyshev's inequality.

Take $E_n = \{x||f_n - f| > \epsilon_n\}$, we can find a $\epsilon_n \to 0$ and subsequence s.t. $m(E_{n(k)}) < \frac{1}{2^k}$.

With the Boreal-Cantalli Lemma (11.6.1), we know $m(\limsup E_n) = 0$. This is equivalent to $f_n \to_{a.e.} f$. □

Lemma 11.6.1. Boreal-Cantalli Lemma: If $\sum m(E_i) \leq \infty$, $m(\limsup E_i) = 0$.

Proof. It is easy because $\limsup E_i = \cup_{j=i}^{\infty} E_j$, so $m(\limsup E_i) \leq \sum_{j=i}^{\infty} E_j$. The latter converge to zero. □