Chapter 5: Differential Form
from Real Mathematical Analysis

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1 Definition of Differential form

In physics, we have the problem of potential energy. For example, suppose a butterfly flies with the cure $C(t) \in X \times Y$, and Y is the horizontal axle. Assume that the Energy Field is F. Then, what is the butterfly's increasing of potential energy?

Suppose $t \in [0, 1]$. We know that we can write this question as

$$\int_0^1 F(x(t), y(t)) \frac{dy(t)}{dt} dt$$

. Since this kind of problem is frequently seen (maybe?, I don't know, I study computer science.), we might rewrite it as

$$\int_C F(C) dy$$

Means integral on C, and y is the "direction" we care.

1.1 Some Terms

In the previous example, C is a curve and y is the direction we care. However, we may not only care a curve (perhaps a surface), and not only care a direction. Then, what we do?

Definition. K-cell φ on $\mathbb{R}^n$: We assume φ is smooth, I^k is the cube $[0, 1]^k$ and φ maps $I^k \to \mathbb{R}^n$

In $k = 1$, it is obviously, a smooth curve.

We define $C_k(\mathbb{R}^n)$ as the set of all k-cell on $\mathbb{R}^n$.

Definition. Differential $k-$form in $\mathbb{R}^n$: Suppose we have a $k-$cell φ on $\mathbb{R}^n$, and we only care some directions $I = \{i_1, i_2., ..., i_k\} \in n$. (Must be a full k). If we weight each φ a value $F : \mathbb{R}^n \to \mathbb{R}$, we define the integral of φ with previous mentioned care terms as

$$\int_\varphi F dy_I = \int_{I^k} F(\varphi(u)) \frac{\partial \varphi_I}{\partial u} du \qquad (1)$$

Here, the $\frac{\partial \varphi_I}{\partial u}$ is the determinant of the Jacobian matrix φ_I over u.

For notation's convenience, We represent $I^k \in \mathbb{R}^k$ as $u = (u_1, u_2, ...u_k)$ (sometimes use x). And, $\varphi : I^k \to \mathbb{R}^n$ as $\varphi((u_1, u_2, ..., u_k)) = (\varphi_1, \varphi_2, ..., \varphi_n)(u) = (y_1, y_2, .., y_k)$

1.2 Reparameterization, Form Name, and Wedge Product

These are fundamental. Just google them.

2 Exterior Differentiation

Actually, I don't really know why we initially do this. I just guess the origin. Perhaps We need exterior differentiation because we also want to know what will happen if the f_I drift away from the y_{N-I} directions.

So we define $d(f_I dy_I)$ as $\sum_{i=1}^{n} \frac{\partial f_I}{\partial y_i} dy_i \wedge dy_I = (df_I) \wedge dy_I$.

2.1 Property of exterior Differentiation

Theorem 2.1. Some property of the exterior Differentiation:

1. d is a linear operator.
2. $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$
3. $d^2 = 0$.

The proof of these theorem are just following from definition.

3 Pushforward and Pullback

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$, $\varphi \in C_k(\mathbb{R}^n)$ is a k-cell and ω is a $k-$form in $\mathbb{R}^m$. Then, if we send the value of φ to $\mathbb{R}^m$ with T, we have a $k-$form on $\mathbb{R}^m$,which is $T \circ \varphi$. We now can calculate the value of $\int_{T\varphi} \omega$. Now, we know the $\int_T \omega$ is a functional of $C_k(R^n)$. We want to find the α s.t.

$$\int_{T\varphi} \omega = \int_{\varphi} \alpha.$$

And we assume T^* as the mapping of ω and $T^*(\omega) = \alpha$.

To avoid the abuse of notation, $T_* : C_k(\mathbb{R}^n) \to C_k(\mathbb{R}^m)$ with $T_*(\varphi) = T \circ \varphi$

For simplicity, assume $I = \{1, 2, ..., k\}$.

Theorem 3.1. $T * (f dz_I) = (f \circ T) dT_1 \wedge dT_2 \wedge ... \wedge dT_k$

Proof. Pick an arbitrary $\varphi : I^k \to R^n$, and $k-$form in $\mathbb{R}^m$ $f_I dz_I$ We have

$$\int_{T*(\varphi)} f_I d\omega = \int_{I^k} f_I \circ T(\varphi(u)) \frac{\partial (T(\varphi(u)))_I}{\partial u} du. \tag{2}$$

We have

$$\frac{\partial (T(\varphi(u)))_I}{\partial u} du = \frac{\partial (T_I(\varphi(u)))}{\partial u} du = \sum_A \{\frac{\partial (T_I)}{\partial y_A}\}_{y=\varphi(u)} \frac{\partial \varphi(u)}{\partial u} du$$

Note that there is a usage of the representation for Cauchy-Binet Formula. Then, by the writing of differential form, we have (2) written this way:

$$T^*(f_I dz_I) = f \circ T \sum_A \frac{\partial(T_I)}{\partial y_A} dy_I$$

For the $f \circ T\ dT_1 \wedge dT_2 ... dT_k$ part, with simple expansion of part, we have

$$dT_1 \wedge dT_2 ... dT_k = (\sum_1^n \frac{\partial T_1}{\partial y_i} dy_i) \wedge ... \wedge (\sum_1^n \frac{\partial T_k}{\partial y_i} dy_i) \tag{3}$$

From the fact that wedge term cannot repeat(or equal to 0), we have (3) equals to :

$$\sum_A \sum_{\sigma(A)} sig(\sigma) \frac{\partial T_1}{\partial y_{\sigma_1}} ... \frac{\partial T_k}{\partial y_{\sigma_k}} dy_A$$

It equals to $\sum_A \frac{\partial T_I}{\partial y_A} dy_A$ by Leibniz formula. With simple comparison we know they are equal. □

Theorem 3.2. $T^*(\alpha \wedge \beta) = T^*(\alpha) \wedge T^*(\beta)$

Proof. This one is simple from the fact that $T^*(a\ d\alpha \wedge b\ d\beta) = (a \circ T)(b \circ T) T^*(d\alpha \wedge d\beta) = T^*(d\alpha) \wedge T^* d(\beta)$. The result is transparent. □

Theorem 3.3. $T^*(d\omega) = dT^*(\omega)$

Proof. Take f as a 0 form.

$$\begin{aligned} T^*(df) &= T^*(\sum_{i=1}^m \frac{\partial f}{\partial z_i} dz) \\ &= \sum_{i=1}^m T^*(\frac{\partial f}{\partial z_i})(dz) \\ &= \sum_{i=1}^m \frac{\partial f}{\partial z_i} \circ T\ (dT_i) \\ &= \sum_{j=1}^n \sum_{i=1}^m \frac{\partial f}{\partial z_i}|_{z=T(y)} \frac{\partial T_i}{\partial y_y} dy_j = d(f \circ T) = dT*(f) \end{aligned}$$

With previous theorem, we know the result holds. □

4 Stock's Formula

Theorem.

$$\int_{\partial M} \omega = \int_M d\omega$$

Before the proof, we need to define the ∂ of the cell M. Intuitively, it is the margin of M

Proof. Here is the step of the proof:

1. Observe the case of $n-1$ form (of $\mathbb{R}^n$) on cube I^k.
2. Generalise what is the ∂.
3. Use pushforward and pullback to calculate the general case.

Here is the extend of each steps:

1. Suppose $\omega = f dx_{-j}$, with dx_{-j} means $dx_1 \wedge ... \wedge dx_{j-1} \wedge dx_{j+1} \wedge ... \wedge dx_n$. Then, $d\omega = (-1)^{j+1} \frac{\partial f}{\partial x_j} dx_N$, with $N = \{1, 2, ..., n\}$. So $\int_{I^k} d\omega =$

$$\int_{I^k} (-1)^{j+1} \frac{\partial f}{\partial x_j} dx_1 dx_2 ... dx_n$$
$$= \int_0^1 ... \int_0^1 (-1)^{j+1} f(..., x_{j-1}, 1, x_{j+1}) - f(..., x_{j-1}, 1, x_{j+1}) dx_1 ... dx_{j-1} dx_{j+1} ... dx_n$$

2. From the previous observation, we know that we may assume

$$\partial I^k = \sum_{i=1}^{k} (-1)^{i+1} (x^{j,1} - x^{j,0})$$

With $x \in I^{k-1}$ and $x^{j,k} = (x_1, x_2, ... x_{j-1}, k, x_{j+1}, ..., x_n)$. Similarly, define φ a $k-$form on $\mathbb{R}^n$, $\partial\varphi = \sum_{i=1}^{k} (-1)^{i+1} (\partial(x^{j,1}) - \partial(x^{j,0}))$. We will see the definition is well-defined in next step.

3. Naturally, we can regard the $\varphi \in C^n(\mathbb{R}^m)$ as a pushforward function. $\tau : I^n \to I^n$ an $n-$cell in $\mathbb{R}^n$. If ω is an $n-1$ form in R^m. Then,

$$\int_\varphi d\omega = \int_\tau \varphi^* d\omega = \int_\tau d\varphi^* \omega = \int_{\partial\tau} \varphi^* \omega = \int_{\varphi(\partial\tau)} \omega = \int_{\partial\varphi} \omega \tag{4}$$

Here, the $\phi*$ after the third equation plays the role of pulling $\Omega^{n-1}(R^m) \to \Omega^{n-1}(R^n)$.

□

4.1 Example: Divergent and Curl

Here, f_x means $\frac{\partial f}{\partial x}$.

Theorem 4.1 (Green's Theorem)**.**

$$\int\int_D (g_x - f_y)dxdy = \int_C fdx + gdy$$

C is the boundary of D with positive direction.

Theorem 4.2 (Green's Divergent Theorem)**.**

$$\int\int\int_D div\ F = \int\int_S flux\ F$$

Here, $F =< f^1, f^2, f^3 >$, and $div\ F =< f^1_x, f^2_y, f^3_z >$ and $flux\ F$ is $f^1dydz+f^2dxdz+f^3dxdy$

We may take the F as curl of some function $G = (g1, g2, g3)$, so we have the Stoke's curl theorem.

5 Close and Exact Form

Assume ω is a $k-$form .

Definition. ω is closed is $d\omega = 0$. ω is exact is $\exists\alpha$ s.t.$d\alpha = \omega$.

5.1 Cohomology

Definition. Suppose U is a open set.
$B^k(U)$ is the exact $k-$form on U. $Z^k(U)$ is the closed $k-$form on U.

$$H^k(U) = Z^k(U)/B^k(U)$$

This is the quotient concept from algebra.
What is simply connected domains: https://www.youtube.com/watch?v=9jyKUjbUjSg

5.2 Poincare's Lemma

Theorem. If $U = R^n$, $H^k(U) = \{0\}$

Proof. Here is the step of proof:

1. Our goal is to prove exist a integral operator L for all $\omega = f_I(x)dy_I$ such that $(dL + Ld)\omega = \omega$.

2. Prove exist an operator N on axle t s.t. $\omega' = f_I'(x,t)dt \wedge dy_I$ and $(Nd + Nd)\omega = f(x,1) - f(x,0)dy_I$.

3. Prove with $p(x,t) = tx$, we have Np^* is the L.

Here is the extend of each steps:

1. Obviously, it is a stronger property. Since $d\omega = 0$, so $L\omega$ is the α we want.

2. Suppose $\omega \in \Omega^{k+1}(R^n \times \mathbb{R})$, I is length of $k+1$, J is of length of k,and operator K^t has this property:

$$K^t(\omega) = \begin{cases} 0 & \text{if it is } dy_I\text{, no } dt \text{ in this wedge product} \\ (\int_0^1 f(y,t)dt)dy_J & \text{it is } dt \wedge dy_J \end{cases} \tag{5}$$

   Then, with some calculus, we have that

$$N^t(dw) + dN^t(w) = (f(y,1) - f(y,0))dy_I \tag{6}$$

3. pick $p(y,t) = ty$, so with more calculation, we have

$$L = N \circ p^*$$

□

6 Bouver's Fixed Point Theorem

Theorem. B is a open ball on $\mathbb{R}^n$, suppose f is a continuous map from $B \to B$ Then, B has a fixed point for f.

Proof. Here is the step of proof:

1. Assume f is infinitely differentiable and $|f(x) - x| > \mu$. Construct a T that map each x to ∂B from the direction $[f(x), x]$.

2. Prove the fact that T^* is a zero map.

3. We have a smooth $\varphi : I^n \to B$ with these property:
   (a) $\varphi(\partial I^n) = \partial B$ (b) $\int_{I^n} \frac{\partial \varphi}{\partial u} du > 0$

4. Then we can find a $\alpha = y_1 dy_{-1}$ s.t. $\int_\varphi d\alpha > 0$ but $\int_{\partial\varphi} \alpha = 0$.

Here is the extend of each steps:

1. Since B is closed, from Stone-Weierstrass Theorem, we can construct a F' s.t. $F' - F \leq \mu/2$. Further construct $G = \frac{F'}{1+\mu/2}$, it can map all point in U, a small neighbourhood of B, into B. Obvious, U still has no fixed point under G.
   Here, since no fixed point, and G is smooth, the function T is smooth, too.

2. With inversion function formula, if $(DT)_p$ is invertible in a point t, there is a small (n-) ball in $T(p)$ that is the image of a neighbourhood of p. However, this is absurd as $T(p) \in \partial B$. As a result, $(DT)_p$ is not invertible. As a result, $T^*(w) = 0 \ \forall w \in \Omega^n(U)$.

3. We map the hyper cube $[-1,1]^n$ to the B with ϕ this way: for each $x \in [-1,1]^n$, take $\phi(x) = s(x)x$, here, suppose the longest (positive)length in $(0,x)$ direction is $l(r)$. Then, $s(x) = |r|/|l(r)|$.

   So we take $\varphi : I^k \to \phi(2I^k - 1^n)$, it is smooth, obviously. $\partial\varphi = \partial B$, which is transparently, too. As for the $\int_{I^k} \frac{\partial \varphi}{\partial u} du$, we know it is the volume of the ball (since it is orientation-preserved, no need to take its absolute value). It satisfies our requirement.

4. We take $\alpha = x_1 dx_2 \wedge ...dx_n$, do $d\alpha = dx_N$. Then from 3. we have

$$\int_\varphi d\alpha > 0$$

   However, since T on $\partial\varphi$ is a identity map, and by 2., we also have

$$\int_\varphi d\alpha = \int_{\partial\varphi} \alpha = \int_{T_*(\partial\varphi)} \alpha = \int_{\partial\varphi} T^*(\alpha) = 0.$$

So prove by contradiction, the result follows. □